# Chapter 1

## Introduction

I've always been interested in nonlinear effects in electronic circuits. I did my thesis on TRAPATT diode oscillators; TRAPATTs are extremely nonlinear devices. Anyway, it is common in the circuit design technology to use the Volterra series for circuits and devices with 'memory' , operating under periodic boundary conditions. Under these conditions the Fourier series is the appropriate vehicle for analysis and design.

But I've also been interested in the Laplace transform as a means of nonlinear circuit and device analysis. The Laplace transform is a more general transform than Fourier. In fact, the Fourier transform is generally treated as a special case of the Laplace transform. For a long time, I thought that the Laplace transform was only useable for strictly linear systems. How wrong I was. Recently, as I was delving into the theoretical basis of the Volterra series, I happened upon the quotation below. It's from eom.springer.de/V/v096870.htm:

“A non-linear input-output dynamical system with input $u(t)$ and output $y(t)$ gives rise to a Volterra series of the form

$$y(t)=\int_{-\infty}^{\infty} h_1(\tau_1)u(t-\tau_1)d\tau_1$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2+....$$

$$+\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n)u(t-\tau_1)u(t-\tau_2)...u(t-\tau_n)d\tau_1...d\tau_n \quad ... \text{“} \quad (1.1)$$

This was exactly what I was looking for; namely, a mathematically rigorous method of nonlinear circuit and device analysis that could be incorporated with the Laplace transform. So, to proceed.

## 1.0 Derivations and Proofs of the utility of (1.1)

Now, in a electronic circuit, $u(t-\tau)$ would be the stimulus (i.e., input voltage) and $y(t)$ would be the response (i.e., output current, $i(t)$ , and $h_n(\tau_1..\tau_n)$ would be the nth order transfer function of the circuit under consideration. BTW, for convenience, I have moved the lower limit to 0; this will be the case from here on out.

Now, the first term in (1.1) looks like the convolution integral; and it is. Additionally, if $h_1(\tau_1)$ is linear in $\tau_1$ , then this first term is a linear operation.

Now let us rewrite the second term on (1.1), and call it $i_2(t)$

$$i_2(t)=\int_0^{\infty} u(t-\tau_2)d\tau_2\int_0^{\infty} h_2(\tau_{1,}\tau_2)u(t-\tau_1)d\tau_1 \quad (1.2)$$

For this term, if $h_2(\tau_1,\tau_2)$ is linear in both $\tau_1$ and $\tau_2$ , then the $\tau_1$ integral is linear, and importantly then, the $\tau_2$ integral is also linear. And thus (1.2) is a linear response function. I would look at it as a double convolution, each of which is linear. More accurately, I would think of (1.2) as a convolution of a convolution. To make this connection, look at the following,

$g_2(t,\tau_2)=\int_0^\infty h_2(\tau_{1,}\tau_2)u(t-\tau_1)d\tau_1$ , we see that it is similar to the first term in (1.2) above, a convolution. Then we can rewrite (1.2) as,

$i_2(t)=\int_0^\infty g_2(t,\tau_2)u(t-\tau_2)d\tau_2$ , thus a convolution of a convolution.

Important note: (1.2) is linear in $\tau_2$ , but generally not linear in $t$ .

Furthermore, we can then deduce that if all orders of the transfer function, $h_n(\tau_1..\tau_n)$ , are linear, then we have an nth order convolution, all of which are linear.
So far, so good. Now if we want to work in the Fourier transform domain--as many circuits can be analyzed there--then we can just turn the crank. There are even software tools that will turn the crank for us. But, what I find compelling about our representation here, is this: assuming we can come up with the required nth order transfer function, $h_n(\tau_1..\tau_n)$ , then not only can we work in the Fourier transform space (if we choose to), but we can also work in the Laplace transform domain. This allows analysis of a much broader class of driving function, $u(t-\tau)$ (not just periodic ones, as is the case for Fourier transform analysis).

Let us look at the Laplace transform of $i(t)$
We will prove that the Laplace transform of (1.1) is,

$$I(s)=G_1(s)V(s)+\int_{-i\infty}^{i\infty} ds_1 G_2(s_{1,}s-s_1)V(s_1)V(s-s_1)$$

$$+\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} ds_1 ds_2 G_3(s_1,s_2,s-s_1-s_2)V(s_1)V(s_2)V(s-s_1-s_2)$$

$$+\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} ds_1 ds_2 ds_3 G_3(s_1,s_2,s-s_1-s_2)V(s_1)V(s_2)V(s-s_1-s_2)+... \qquad (1.3)$$

Do notice that I have changed notation:
where: $I(s)=L[i(t)]$ and $V(s)=L[v(t)]$ ; and $G_n(s_{1,}s_{2,}..s_n,s-s_1-s_2...-s_n)$ is the nth order, multidimensional Laplace transform of $g_n(\tau_1..\tau_n)$ . Mathematically,

$I(s)=\int_0^\infty i(t)e^{-st}dt$ , and similarly for $V(s)$ .

And for $G_n(s_{1,}s_{2,}..s_n,s-s_1-s_2...-s_n)$ we have,

$$=\int_0^\infty..\int_0^\infty g_n(t_1,..t_n)e^{-s_1t_1}e^{-s_2t_2}..e^{-(s-s_1-s_2..-s_n)t_n}dt_1..dt_n \qquad (1.4)$$

Now, we need to demonstrate the validity of (1.3). We do this by taking the inverse Laplace transform.

That is, we want to find $L^{-1}[I(s)]=i(t)$ and likewise for $v(t)$ .
Now clearly, the first term on the RHS of (1.3) is the conventional convolution integral, namely

$$L^{-1}[G_1(s)V(s)]=\int g_1(t-\tau)v(t)dt$$

For the higher order terms, however, we need to carry out the derivation. For the 2nd term, we have

$$i_2(t)=\int_{-i\infty}^{i\infty}\exp^{st}ds\int_{-i\infty}^{i\infty}ds_1\,G_2(s_{1,}s-s_1)V(s_1)V(s-s_1) \qquad (1.5)$$

We will expand the functions in s in terms of their respective Laplace transforms, thusly:

where $G_2(s_{1,}s-s_1)=\int_0^{\infty}g_2(t_{1,}t_2)e^{-s_1t_1}e^{-(s-s_1)t_2}dt_1dt_2$ we get,

$$i_2(t)=\int_{-i\infty}^{i\infty}e^{st}ds\int_{i\infty}^{i\infty}ds_1\int_0^{\infty}g_2(t_{1,}t_2)e^{-s_1t_1}e^{-(s-s_1)t_2}dt_1dt_2\int_0^{\infty}v(t_3)e^{-s_1t_3}dt_3\int_0^{\infty}v(t_4)e^{-(s-s_1)t_4}dt_4$$

. (1.6)

We can rearrange (1.6) in the following manner,

$$i_2(t)=\int_{-i\infty}^{i\infty}ds_1\int_0^{\infty}g_2(t_{1,}t_2)e^{-s_1t_1+s_1t_2}dt_1dt_2\int_0^{\infty}v(t_3)e^{-s_1t_3}dt_3\int_0^{\infty}v(t_4)e^{s_1t_4}dt_4\int_{-i\infty}^{i\infty}ds\,e^{s(t-t_2-t_4)}$$

. (1.7)

Now, look at just the last term in (1.7),

$$\int_{-i\infty}^{i\infty}e^{s(t-t_2-t_4)}ds\rightarrow\delta(t-t_2-t_4)$$

We apply this to the adjacent integral (and $t_4\rightarrow t-t_2$ ), which takes it to $v(t-t_2)\exp^{s_1(t-t_2)}$ .

And so, to recap,

$$i_2(t)=\int_{-i\infty}^{i\infty}ds_1\int_0^{\infty}g_2(t_1,t_2)e^{-s(t_1-t_2)}dt_1dt_2\int_0^{\infty}dt_3v(t_3)\exp^{-s_1t_3}v(t-t_2)e^{s_1(t-t_2)} \qquad (1.8)$$

.

And, again, rearranging yields,

$$i_2(t)=\int_0^{\infty} dt_1 \int_0^{\infty} g_2(t_1,t_2)\,v(t-t_2)\,dt_2 \int_0^{\infty} dt_3\, v(t_3) \int_{-i\infty}^{i\infty} ds_1 e^{s_1(t-t_1-t_3)} \qquad (1.9)$$

The last term in (1.9) goes to $\delta(t-t_1-t_3)$ , which takes the adjacent integral, in $t_3$ , to $v(t-t_1)$ .

This yields up our solution, namely

$$i_2(t)=\int dt_1 \int g_2(t_1,t_2)\,v(t-t_2)\,v(t-t_1)\,dt_2 \qquad (1.10)$$

In like manner, the higher order terms can be shown to reduce to our solution; these solutions are not more complicated, just longer. As such, we will forgo them.

## 1.2 Nonlinear Capacitor

The nonlinear capacitor is an interesting and useful device. It is employed in electronic circuits in all kinds of way. Let's look at one such device. Suppose we have the following capacitance versus voltage relationship,

$$c(v)=\frac{c_0}{\sqrt{(\phi-v)}}$$

We can generate a power series solution for this relationship, namely:

$$c(v)=\sum_{n=0}^{\infty} c_{n+1} v^n$$

And given time-varying voltage across the capacitor, $v\rightarrow v(t)$ , the current through the capacitor is,

$$i(t)=c(v)\frac{dv}{dt} \quad \text{, or}$$

$$i(t)=(c_1+c_2v^1+c_3v^2+c_4v^3+\ldots)\frac{dv}{dt} \quad \text{, or}$$

$$i(t)=c_1\frac{dv}{dt}+c_2\frac{1}{2}\frac{dv^2}{dt}+c_3\frac{1}{3}\frac{dv^3}{dt}+\ldots \qquad (1.11)$$

The first term in (1.11) above is the expected linear term, no problem there.
The remaining terms are all nonlinear in voltage, so we need to do something to them in order to generate the appropriate $g_n(\tau_1..\tau_n)$ 's. Let's take the second term. This would be the $i_2(t)$ in our notation. And we want to find a good $g_2(t_1,t_2)$ . Recall from (1.3) that in the Laplace transform space, we can separate the $v^2$ into $V(s_1)V(s-s_1)$ . Likewise we want our $g_2(t_1,t_2)$ to look like $G_2(s_1, s-s_1)$ . To do this we can think of $v^2$ as being the product $v_1 v_2$ ( where $v_1=v(t_1)$ and $v_2=v(t_2)$ ).

For the capacitor then, the 2nd order term goes like:

$$i_2(t) = c_2 \frac{1}{2}\frac{dv^2}{dt} \rightarrow c_2 \frac{1}{2} d\frac{v_1 v_2}{dt} = \frac{c_2}{2}(v_2 \frac{dv_1}{dt_1} + v_1 \frac{dv_2}{dt_2})$$

Now we take the double Laplace transform of this 2nd order term,

$$G_2(s_{1,} s_2) = \frac{c_2}{2}\int_0^\infty \int_0^\infty (v_2 \frac{dv_1}{dt_1} + v_1 \frac{dv_2}{dt_2}) e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2$$

Rearranging,

$$= \frac{c_2}{2}\int_0^\infty \exp^{-s_2 t_2} dt_2 \int_0^\infty (v_2 \frac{dv_1}{dt_1} + v_1 \frac{dv_2}{dt_2}) e^{-s_1 t_1} dt_1$$

The integral in $t_1$ goes to,

$$v_2 s_1 V(s_1) + V(s_1)\frac{dv_2}{dt_2}$$

And likewise, the integral in $t_2$ goes to,

$$V(s_2) s_1 V(s_1) + V(s_1) s_2 V(s_2)$$

We can change variables, $s_2 \rightarrow s - s_1$ and the 2nd order term goes to

$$G_2(s_{1,} s - s_1) = \frac{c_2}{2} s V(s_1) V(s - s_1) \qquad (1.12)$$

We should be able to generalize the higher order terms as,

$$G_n(s_{1,} s_2 ... - s - s_1 - s_2 .. s - s_n) = \frac{c_n}{n} s V(s_1) V(s_2) ... V(s - s_1 - s_2 ... - s_n) \qquad (1.13)$$

Again, the above generalization is not more complex, just long, so we will leave it to the reader of lefttothereader....dude.

We can now use this stuff and examine a real circuit in the next chapter.